Fall 2023 Math 1B Final Review Sheet (Haiman)

Basic Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}a^x = a^x \ln a$$
$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Basic Integrals

Don't forget $+C$ on all of these!

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$
$$\int \frac{1}{x}\,dx = \ln|x|$$
$$\int e^x\,dx = e^x$$
$$\int a^x\,dx = \frac{a^x}{\ln a}$$
$$\int \sin x\,dx = -\cos x$$
$$\int \cos x\,dx = \sin x$$
$$\int \sec^2 x\,dx = \tan x$$
$$\int \csc^2 x\,dx = -\cot x$$
$$\int \sec x \tan x\,dx = \sec x$$
$$\int \csc x \cot x\,dx = -\csc x$$
$$\int \tan x\,dx = \ln|\sec x|$$
$$\int \cot x\,dx = \ln|\sin x|$$
$$\int \sec x\,dx = \ln|\sec x + \tan x|$$
$$\int \csc x\,dx = \ln|\csc x - \cot x|$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x$$
$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x$$

Trigonometric Identities

Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$

Half-angle identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Double-angle identities:

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

Sum-and-difference identities:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Product identities:

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

Substitution

Indefinite:

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du.$$

Definite:

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Integration by Parts

Indefinite:
$$\int u\,dv = uv - \int v\,du.$$

Definite:
$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

To pick u, use the LIATE rule:

1. Logarithmic (e.g. $\ln x$)
2. Inverse trigonometric (e.g. $\arcsin x$)
3. Algebraic (e.g. x^2)
4. Trigonometric (e.g. $\sin x$)
5. Exponential (e.g. e^x)

The function higher on the list should be u. The remaining factor should be dv. *Warning: LIATE is only a guideline, not a rule. It will not work every time.*

Trigonometric Substitution

Integrals involving the following expressions can usually be simplified with trigonometric substitution:

Expression	Substitution	Trig Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Note: there need not be a square root in order to apply trig substitution. It can apply to more general integrals.

Terms like $\sin(\arctan(x/a))$ can be simplified by drawing a triangle and using the Pythagorean theorem.

Trigonometric Integrals

Strategy for $\int \sin^m x \cos^n x\,dx$:

1. If n is odd, save one factor of $\cos x$, rewrite in terms of $\sin x$, and substitute $u = \sin x$.

   Example:
   $$\begin{aligned}\int \sin^2 x\cos^5 x\,dx &= \int \sin^2 x\cos^4 x\cos x\,dx\\ &= \int \sin^2 x(1-\sin^2 x)^2\cos x\,dx\\ &= \int u^2(1-u^2)^2\,du.\end{aligned}$$

2. If m is odd, save one factor of $\sin x$, rewrite in terms of $\cos x$, and substitute $u = \cos x$.
3. If both n and m are even, then use the *half-angle identities.*

Strategy for $\int \tan^m x \sec^n x\,dx$:

1. If n is even, then save a factor of $\sec^2 x$ and let $u = \tan x$.
2. If m is odd, then save a factor of $\sec x\tan x$ and let $u = \sec x$.
3. Otherwise, other strategies are needed. Use trig identities to help.

For $\int \sin(ax)\cos(bx)\,dx$, $\int \sin(ax)\sin(bx)\,dx$, and $\int \cos(ax)\cos(bx)\,dx$, use the *product identities* to write as a sum.

Integration by Partial Fractions

The method of partial fractions is useful to solve integrals involving *rational functions* $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials.

1. If the degree of the numerator is greater than or equal to the degree of the denominator, divide the numerator by the denominator.
2. Factor the denominator. Identify the linear and irreducible quadratic factors.
3. Perform partial fraction decomposition.
4. Integrate each term separately.

Cases for partial fraction decomposition:

1. Distinct linear factors: use A, B, C, etc. for numerators.

   Example:
   $$\frac{x^2+2x+1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

2. Repeated linear factors: add a separate term for each power.

   Example:
   $$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

3. Irreducible quadratic factors: use $Ax+B$, $Cx+D$, etc. for numerators.

   Example:
   $$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

4. Repeated irreducible quadratic factors: add a separate term for each power.

   Example:
   $$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Common Algebraic Tricks

Some integrals require algebraic manipulation before solving with one of the standard techniques. Here are a few common cases:

- Rationalizing substitutions: perform a substitution to turn a nonrational function into a rational function.

  Example: Using $u = \sqrt{x+4}$,
  $$\int \frac{\sqrt{x+4}}{x}\,dx = 2\int \frac{u^2}{u^2-4}\,du.$$
  This integral can now be completed using trig sub or partial fractions.

- Completing the square: eliminate the linear term in a quadratic.
  $$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right).$$
  Example:
  $$\int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{(x-1)^2+1}\,dx$$

- Multiplying by the conjugate: multiplying the numerator and denominator by a conjugate can be useful for simplification.

  Example:
  $$\begin{aligned}&\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx \\ &= \int \frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\,dx \\ &= \int \frac{\sqrt{x+1}-\sqrt{x}}{(x+1)-x}\,dx \\ &= \int (\sqrt{x+1}-\sqrt{x})\,dx\end{aligned}$$

- Substituting $\sqrt{x}$: if the integrand contains $\sqrt{x}$, then $u = \sqrt{x}$ can often be a useful substitution.

  Example:
  $$\int e^{\sqrt{x}}\,dx = 2\int ue^u\,du.$$
  Now use integration by parts.

Numerical Integration

In each method, $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 0, 1, 2, \ldots, n$. The **error** E of an approximate integral is the difference between the true and approximated value of the integral.

Midpoint rule:
$$\begin{aligned}M_n = \Delta x\Bigg[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) \\ + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\Bigg]\end{aligned}$$
$$|E_M| \le \frac{K(b-a)^3}{24n^2}, \qquad K = \max_{[a,b]} |f''(x)|$$

Trapezoidal rule:
$$T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$
$$|E_T| \le \frac{K(b-a)^3}{12n^2}, \qquad K = \max_{[a,b]} |f''(x)|.$$

Simpson's rule:
$$\begin{aligned}S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) \\ + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]\end{aligned}$$
$$|E_S| \le \frac{L(b-a)^5}{180n^4}, \qquad L = \max_{[a,b]} |f^{(4)}(x)|$$

Arc Length

The **arc length** of the curve $y = f(x)$ from $x = a$ to $x = b$ is
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

Improper Integrals

Two types of improper integrals:

1. Infinite intervals: endpoints at $\pm\infty$.
$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

2. Discontinuities: if $f(x)$ is discontinuous at b,
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

An improper integral is **convergent** if the limit(s) exist. Otherwise, it is **divergent**.

Comparison Theorem: Suppose that f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

- If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
- If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Center of Mass

The **center of mass**, or **centroid**, of N masses $m_1, m_2, \ldots, m_n$ at points $x_1, x_2, \ldots, x_n$, is given by
$$\overline{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}$$

For a plate of uniform density between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$, the center of mass is
$$\overline{x} = \frac{M_y}{A} = \frac{\int_a^b x[f(x) - g(x)]\,dx}{\int_a^b f(x)\,dx}$$
$$\overline{y} = \frac{M_x}{A} = \frac{\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx}{\int_a^b f(x)\,dx}$$

Theorem of Pappus: Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$.

Sequences

A **sequence** is an ordered list of numbers:

$$\{a_n\}_{n=1}^{\infty} = \{a_n\} = a_1,\, a_2,\, a_3,\, \ldots$$

A sequence $\{a_n\}$ has a *limit* L if a_n can get arbitrarily close to L as n gets large. If $\lim_{n\to\infty} a_n$ exists, we say it is **convergent**. Otherwise, it is **divergent**.

A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. It is **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. The sequence is **monotonic** if it is either increasing or decreasing.

A sequence $\{a_n\}$ is **bounded above** if there exists a number M such that $a_n \leq M$ for all $n \geq 1$. Likewise, it is **bounded below** if there exists a number m such that $a_n \geq m$ for all $n \geq 1$. If it is bounded above and below, then it is **bounded**.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent and c is constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad (\text{if } \lim_{n\to\infty} b_n \neq 0)$$

$$\lim_{n\to\infty}(a_n^p) = \left[\lim_{n\to\infty} a_n\right]^p \quad (\text{if } p > 0 \text{ and } a_n > 0)$$

Convergence Theorems for Sequences

These theorems are useful for proving the convergence (and corresponding limit) of more complicated sequences.

- If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n\to\infty} a_n = L$.
- (**Squeeze Theorem**) If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
- If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
- If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$,
- (**Monotonic Sequence Theorem**) Every bounded, monotonic sequence is convergent.

Series

A **series**, denoted $\sum a_n$, is an infinite sum of the terms a_n. A **partial sum** of the series is defined as

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_n.$$

A series is **convergent** if $\lim_{n\to\infty} s_n = s$. Otherwise, it is **divergent**.

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent. A series is **conditionally convergent** if it is convergent but not absolutely convergent.

Basic Properties: if $\sum a_n$ and $\sum b_n$ are convergent,

1. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
2. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
3. $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Common Series Examples

Geometric series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

The nth partial sum is

$$s_n = \frac{a(1-r^n)}{1-r}.$$

The geometric series converges for $|r| < 1$ and diverges for $|r| \geq 1$. If $|r| < 1$, the limit is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad (|r| < 1).$$

Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The harmonic series diverges.

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

The p-series converges for $p > 1$ and diverges for $p \leq 1$. When $p = 1$, this is the harmonic series.

Other series

These series are useful when using comparison tests.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \quad (\text{converges as telescoping series})$$

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} \quad (\text{converges by alternating series test})$$

$$\sum_{n=1}^{\infty} \frac{1}{n!} = e \quad (\text{converges by ratio test})$$

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \quad (\text{converges by root test})$$

Divergence Test

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: if $\lim_{n\to\infty} a_n = 0$, then this test is inconclusive!

Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then,

- If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: don't forget about the continuous, positive, and decreasing assumptions. The integral/series need not start at 1: you can start at $n = 2$ or later if needed.

Remainder Estimate for the Integral Test: if $a_n = f(n)$ satisfies the conditions for convergence in the integral test, then the remainder $R_n = s - s_n$ can be estimated as

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Comparison Tests

Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is finite and $c > 0$, then either both series converge or both diverge.

Note: don't forget the positive assumption.

Alternating Series Test

If we can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n,$$

where b_n is positive, decreasing, and $\lim_{n\to\infty} b_n = 0$, then $\sum a_n$ converges.

Notes:

- *$(-1)^n$ and $(-1)^{n+1}$ are valid. $(-1)^{2n}$ is not.*
- $\cos(\pi n) = (-1)^n$.

Alternating Series Estimation Theorem: For a series that satisfies the alternating series test, the remainder $R_n = s - s_n$ is bounded by the next term:

$$|R_n| \leq b_{n+1}.$$

Ratio and Root Tests

Ratio Test:

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum a_n$ is divergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the test is inconclusive.

Root Test:

- If $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$, then $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} |a_n|^{1/n} = L > 1$, then $\sum a_n$ is divergent.
- If $\lim_{n\to\infty} |a_n|^{1/n} = L = 1$, then the test is inconclusive.

Notes:

- *Don't forget to take the absolute value.*
- *For ratio test, look for factorials. For root test, look for nth powers.*

Power Series

A **power series** centered at $x = a$ is of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

The c_n's are the **coefficients** of the series.

Convergence of Power Series: The **radius of convergence** is a number R such that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. If the series converges only when $x = a$, then $R = 0$. If the series converges for all x, then $R = \infty$. The **interval of convergence** consists of all values of x for which the series converges; this includes the endpoints.

To find the interval of convergence, first use the Ratio or Root Test to determine the radius of convergence R. Then, solve for the endpoints using a different convergence test.

Function Representations: use algebraic manipulations (and derivatives/integrals) to switch between functions and their series representations.

Example:

$$\frac{1}{1+3x^2} = \frac{1}{1-(-3x^2)} = \sum_{n=0}^{\infty} (-3x^2)^n = \sum_{n=0}^{\infty} (-3)^n x^{2n}$$

Differentiation and Integration: use the power rule term-by-term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radius of convergence remains the same under differentation and integration, but the convergence of the endpoints can change.

Taylor Series

The **Taylor series** of a function f centered at $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

The **Maclaurin series** is the Taylor series centered at zero. The **Taylor polynomial** $T_n(x)$ of degree n is the partial sum of the Taylor series up to the degree n term.

Taylor's Inequality: if $|f^{(n+1)}(x)| \le M$ for $|x-a| < r$, then the remainder $R_n(x) = f(x) - T_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad (\text{for } |x-a| < r)$$

A function $f(x)$ is **analytic** on $(a-r, a+r)$ if it converges to its Taylor series on $(a-r, a+r)$. To show $f(x)$ is analytic, show $\lim_{n\to\infty} |R_n(x)| = 0$ using Taylor's Inequality.

Common Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{k} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n \qquad R = 1$$

Differential Equations

A **differential equation** contains an unknown function and one or more of its derivatives. Its **order** is the highest derivative that occurs.

The **general solution** to a differential equation is a family of functions containing one or more arbitrary constants. An **initial-value problem** specifies an initial condition to solve for these constants and obtain one solution.

For first-order differential equations, the slope can be plotted at each point. This is called a **direction (slope) field**.

Euler's Method

Consider the initial-value problem

$$y'(x) = F(x, y), \; y(x_0) = y_0.$$

Euler's Method says that the solution $y(x)$ can be approximated as

$$x_n = x_{n-1} + h$$
$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

where h is the **step size**.

Separable Equations

Separable equations take the form

$$\frac{dy}{dx} = g(x)f(y).$$

To solve these, move the x and y terms to different sides and integrate:

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx.$$

Then, solve for $y(x)$.

Note: don't forget $+C$ after the integration step.

First-Order Linear Equations

First-order linear equations take the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

To solve these, multiply both sides by the **integrating factor**

$$I(x) = e^{\int P(x)\,dx}.$$

Then, integrate both sides and solve for $y(x)$.

Note: don't forget $+C$ after the integration step.

Modeling with Differential Equations

Natural Growth: For a population size P, relative growth rate k, and initial population size P_0,

$$\frac{dP}{dt} = kP, \; P(0) = P_0.$$

The solution is

$$P(t) = P_0 e^{kt}.$$

Logistic Growth: For a population size P, relative growth rate k, carrying capacity M, and initial population size P_0,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \; P(0) = P_0.$$

The solution is

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \; A = \frac{M - P_0}{P_0}.$$

Mixing Problems: The general model is

(change in amount) = (rate in) − (rate out)

Predator-Prey: For a prey population R and predator population W, the Lotka-Volterra model is

$$\frac{dR}{dt} = kR - aRW$$
$$\frac{dW}{dt} = -rW + bRW.$$

Equilibrium solutions occur when $\frac{dR}{dt} = \frac{dW}{dt} = 0$.

Second-Order Linear Differential Equations

A **constant-coefficient, second-order, linear differential equation** takes the form

$$ay'' + by' + cy = g(x).$$

It is **homogeneous** if $g(x) = 0$ and **nonhomogeneous** if $g(x) \neq 0$.

An **initial-value problem** specifies $y(x_0)$ and $y'(x_0)$ at some point x_0. A **boundary-value problem** specifies $y(x_0)$ and $y(x_1)$ at two different points x_0 and x_1.

Homogeneous case:

$$ay'' + by' + cy = 0.$$

To solve, find roots of the **auxiliary equation**

$$ar^2 + br + c = 0.$$

Three cases:

1. Distinct real roots r_1 and r_2:
$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$
2. Repeated real roots $r = r_1 = r_2$:
$$y(x) = C_1 e^{rx} + C_2 x e^{rx}.$$
3. Complex roots $r = \alpha \pm \beta i$:
$$y(x) = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x.$$

Nonhomogeneous case:

$$ay'' + by' + cy = g(x).$$

To solve, first find the **homogeneous solution** $y_h(x)$ using the method above. Then, find the **particular solution** $y_p(x)$ using undetermined coefficients or variation of parameters. The general solution is

$$y(x) = y_h(x) + y_p(x).$$

Method of Undetermined Coefficients

For the **method of undetermined coefficients**, we guess the form of $y_p(x)$. There are three key cases:

1. Polynomials: if $g(x)$ is a polynomial, guess a general polynomial of the same degree:
$$g(x) = 3x^2 \implies y_p(x) = Ax^2 + Bx + C.$$
2. Exponentials: if $g(x)$ is an exponential, guess the same exponential with an unknown coefficient:
$$g(x) = 2e^{-4x} \implies y_p(x) = Ae^{-4x}$$
3. Sine/cosine: if $g(x)$ is a sine or cosine, guess a sine and cosine together:
$$g(x) = \cos(2x) \implies y_p(x) = A\cos(2x) + B\sin(2x)$$

If $g(x)$ is a sum, treat each term separately:

$$g(x) = e^{-3x} + \cos(2x)$$
$$\implies y_p(x) = Ae^{-3x} + B\cos(2x) + C\sin(2x)$$

If $g(x)$ is a product, combine the guesses together:

$$g(x) = x\cos(2x)$$
$$\implies y_p(x) = (Ax + B)\cos(2x) + (Cx + D)\sin(2x)$$

If a term in your guess conflicts with the homogeneous solution $y_h(x)$, "boost" $y_p(x)$ by x or x^2:

$$y_h(x) = C_1\cos(2x) + C_2\sin(2x),\ g(x) = \cos(2x)$$
$$\implies y_p(x) = Ax\cos(2x) + Bx\sin(2x)$$

After making a guess for $y_p(x)$, plug into the differential equation, group like terms together, and solve for the undetermined coefficients.

Variation of Parameters

For the **method of variation of parameters**, given the homogeneous solution

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x),$$

guess a particular solution

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x).$$

To solve for $u_1(x)$ and $u_2(x)$, solve the system

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0, \\ a y_p'' + b y_p' + c y_p = g(x). \end{cases}$$

This simplifies down to

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0, \\ a(u_1' y_1' + u_2' y_2') = g(x). \end{cases}$$

Isolate u_1' and u_2' and integrate both sides to solve for $u_1(x)$ and $u_2(x)$.

Vibrating Spring

The **spring force** with **spring constant** k is

$$F_{spring} = -kx.$$

The **damping force** with **damping constant** c is

$$F_{damping} = -c\frac{dx}{dt}.$$

Newton's Second Law yields the equation of motion:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

If $c^2 - 4km > 0$, the spring is **overdamped**. If $c^2 - 4km = 0$, the spring is **critically damped**. If $c^2 - 4km < 0$, the spring is **underdamped**. The **period** of oscillations for the underdamped case is

$$T = \frac{2\pi}{\beta}.$$

An **external force** $F(t)$ is inserted as a nonhomogeneous term in the equation of motion. **Resonance** occurs when $F(t)$ has the same frequency as the homogeneous solution.

Series Solutions

A **series solution** to a differential equation takes the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Steps for solving:

1. Find $y'(x)$ and $y''(x)$ and plug into the differential equation. If there is a nonhomogeneous part, write its equivalent series representation.
2. Match the degree and starting indices of each series and solve for the **recursion relation** of the coefficients c_n.
3. Solve for the general term c_n. If the differential equation is first-order, leave c_0 undetermined. If the differential equation is second-order, leave c_0 and c_1 undetermined.
4. Write $y(x)$ using c_n. You may need to split the series into several series for even/odd values of n, etc.
5. For initial-value problems, solve for c_0 and/or c_1 if necessary.