Spring 2024 Math 1B Final Review Sheet (Prof. Paulin / Troy Tsubota)

Basic Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}a^x = a^x \ln a$$
$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Basic Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \csc x \cot x\,dx = -\csc x + C$$
$$\int \tan x\,dx = \ln|\sec x| + C$$
$$\int \cot x\,dx = \ln|\sin x| + C$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$
$$\int \frac{1}{\sqrt{k^2-x^2}}\,dx = \arcsin\frac{x}{k} + C$$
$$\int \frac{1}{k^2+x^2}\,dx = \frac{1}{k}\arctan\frac{x}{k} + C$$
$$\int \frac{1}{x\sqrt{x^2-k^2}}\,dx = \frac{1}{k}\operatorname{arcsec}\frac{x}{k} + C$$

Trigonometric Identities

Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$

The identities below would be provided on an exam but are useful to know how to apply.

Half-angle identities:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

Double-angle identities:

$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

Substitution

Indefinite:

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du.$$

Definite:

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Notes:

- *DO NOT MIX x's AND u's IN THE SAME INTEGRAL.*
- *For definite integrals, be careful with your bounds.*

Integration by Parts

Indefinite:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Definite:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx.$$

To pick $f(x)$, use the LIATE rule:

1. Logarithmic (e.g. $\ln x$)
2. Inverse trigonometric (e.g. $\arcsin x$)
3. Algebraic (e.g. x^2)
4. Trigonometric (e.g. $\sin x$)
5. Exponential (e.g. e^x)

The function higher on the list should be $f(x)$. The remaining factor should be $g'(x)$.

Notes:

- *LIATE is only a guideline, not a rule. It will not work every time.*
- *The "hidden 1" often shows up in logarithmic and inverse trig integrals.*
- *You may need to use IBP several times. Study* $\int x^2 e^x\,dx$ *and* $\int e^x \sin x\,dx$ *as representative examples.*
- *Study the example* $\int \frac{1}{(1+x^2)^2}\,dx$*. Prof. Paulin really likes this one.*

Integration of Rational Functions

The method of partial fractions is useful to solve integrals involving *rational functions* $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials.

1. If the degree of the numerator is greater than or equal to the degree of the denominator, divide the numerator by the denominator.
2. Factor the denominator. Identify the linear and irreducible quadratic factors.
3. Perform partial fraction decomposition.
4. Integrate each term separately.

Cases for partial fraction decomposition:

1. Distinct linear factors: use A, B, C, etc. for numerators.

   Example:

   $$\frac{x^2+2x+1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

2. Repeated linear factors: add a separate term for each power.

   Example:

   $$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

3. Irreducible quadratic factors: use $Ax+B$, $Cx+D$, etc. for numerators.

   Example:

   $$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

4. Repeated irreducible quadratic factors: add a separate term for each power.

   Example:

   $$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Trigonometric Integrals

Strategy for $\int \sin^a x \cos^b x\,dx$:

1. If a is odd, use $u = \cos x$.
2. If b is odd, use $u = \sin x$.

   Example:

   $$\begin{aligned}\int \sin^2 x \cos^5 x\,dx &= \int \sin^2 x \cos^4 x \cos x\,dx \\ &= \int \sin^2 x (1-\sin^2 x)^2 \cos x\,dx \\ &= \int u^2(1-u^2)^2\,du \\ &= \int u^2(1-2u^2+u^4)\,du \\ &= \int u^2 - 2u^4 + u^6\,du.\end{aligned}$$

3. If $-(a+b) \geq 0$, rewrite in terms of $\tan x$ and $\sec x$ and use $u = \tan x$.

   Example:

   $$\begin{aligned}\int \sin^2 x \cos^{-4} x\,dx &= \int \tan^2 x \sec^2 x\,dx \\ &= \int u^2\,du.\end{aligned}$$

4. If $-(a+b) < 0$, then use the double-angle formula or another trig identity.

   Example:

   $$\begin{aligned}\int \cos^2 x\,dx &= \int \frac{1}{2} + \frac{1}{2}\cos(2x)\,dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.\end{aligned}$$

Notes:

- *If a and b are both odd, just pick one of the options.*

Trigonometric Substitution

Integrals involving the following expressions can usually be simplified with trigonometric substitution:

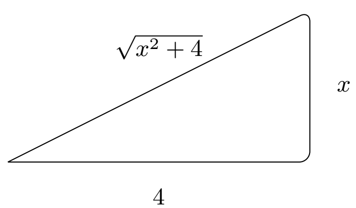
Expression	Substitution	Trig Identity
$\sqrt{k^2 - x^2}$	$x = k\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{k^2 + x^2}$	$x = k\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - k^2}$	$x = k\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

After trig substitution, you will always get a *trig integral* that you can solve using trig integral techniques.

Inverse trig functions inside trig functions can be simplified by drawing a right triangle and using the Pythagorean theorem.

Example:

$$\sin(\arctan(x/4)) = \frac{x}{\sqrt{x^2+4}}.$$



Notes:

- *Do not mix x's and θ's in the same integral.*
- *There need not be a square root in order to apply trig substitution. It can apply to more general integrals.*
- *You may need to complete the square. See "Common Algebraic Tricks."*
- *If there is a leading coefficient in front of x^2 (e.g. $\sqrt{25x^2-4}$), factor it out.*

Common Algebraic Tricks

Some integrals require algebraic manipulation before solving with one of the standard techniques. Here are a few common cases:

- Completing the square: eliminate the linear term in a quadratic.

  $$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

  Example: Using $u = x - 1$,

  $$\begin{aligned}\int \frac{x^2}{x^2-2x+2}\,dx &= \int \frac{x^2}{(x-1)^2+1}\,dx \\ &= \int \frac{(u+1)^2}{u^2+1}\,du \\ &= \int \frac{u^2}{u^2+1}\,du + \int \frac{2u}{u^2+1}\,du + \int \frac{1}{u^2+1}\,du.\end{aligned}$$

  Now complete each integral separately.

- Multiplying by the conjugate: multiplying the numerator and denominator by a conjugate can be useful for simplification.

  Example:

  $$\begin{aligned}\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx &= \int \frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\,dx \\ &= \int \frac{\sqrt{x+1}-\sqrt{x}}{(x+1)-x}\,dx \\ &= \int (\sqrt{x+1}-\sqrt{x})\,dx\end{aligned}$$

- Rationalizing substitutions: perform a substitution to turn a nonrational function into a rational function.

  Example: Using $u = \sqrt{x+4}$,

  $$\int \frac{\sqrt{x+4}}{x}\,dx = 2\int \frac{u^2}{u^2-4}\,du.$$

  This integral can now be completed using trig sub or partial fractions.

Approximate Integration

In each method, $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 0, 1, 2, \ldots, n$. The **error** of an approximate integral is the difference between the true and approximated value of the integral:

$$(\text{error}) = (\text{true value}) - (\text{approximate value})$$

Midpoint rule:

$$M_n = \Delta x\left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

$$|E_M| \le \frac{K(b-a)^3}{24n^2}, \qquad K \ge \max_{[a,b]} |f''(x)|$$

Trapezoidal rule:

$$T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$|E_T| \le \frac{K(b-a)^3}{12n^2}, \qquad K \ge \max_{[a,b]} |f''(x)|.$$

Simpson's rule:

$$S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

$$|E_S| \le \frac{K(b-a)^5}{180n^4}, \qquad K \ge \max_{[a,b]} |f^{(4)}(x)|.$$

Calculating K: Break down $|f''(x)|$ or $|f^{(4)}(x)|$ using these two rules:

- $|A + B| \le |A| + |B|$ ("triangle inequality")
- $|A \cdot B| = |A| \cdot |B|$

Then maximize each term *separately.*

Example: On the domain $[1, 2]$,

$$\begin{aligned}|3x - 4x^3\sin(x) + e^{-x}| &\le |3x| + |4x^3\sin(x)| + |e^{-x}| \\ &\le 3|x| + 4|x^3||\sin(x)| + |e^{-x}| \\ &\le 3\cdot 2 + 4\cdot 2^3 \cdot 1 + e^{-1}.\end{aligned}$$

Improper Integrals

Two types of improper integrals:

1. Infinite intervals: endpoints at $\pm\infty$.

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx.$$

2. Discontinuities: if $f(x)$ is discontinuous at b,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.$$

An improper integral is **convergent** if the limit(s) exist. Otherwise, it is **divergent**.

Comparison Theorem: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
- If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Key cases for the Comparison Theorem:

- $\int_1^\infty 1/x^p\,dx$ converges if $p > 1$ and diverges if $p \leq 1$.
- $\int_0^1 1/x^p\,dx$ converges if $p < 1$ and diverges if $p \geq 1$.
- $\int_{-\infty}^0 b^x\,dx$ $(b > 1)$ converges.

Notes:

- *The discontinuity may be inside your interval. Double check and split up the integral if needed. If one part diverges, then the entire integral diverges.*

Sequences

A **sequence** is an ordered list of numbers:

$$\{a_n\}_{n=1}^\infty = \{a_n\} = a_1,\, a_2,\, a_3,\, \ldots$$

A sequence $\{a_n\}$ has a *limit* L if a_n can get arbitrarily close to L as n gets large. If $\lim_{n\to\infty} a_n$ exists, we say it is **convergent**. Otherwise, it is **divergent**.

A sequence $\{a_n\}$ is **bounded above** if there exists a number M such that $a_n \leq M$ for all $n \geq 1$. Likewise, it is **bounded below** if there exists a number m such that $a_n \geq m$ for all $n \geq 1$. If it is bounded above and below, then it is **bounded**.

Series

A **series**, denoted $\sum a_n$, is an infinite sum of the terms a_n. A **partial sum** of the series is defined as

$$s_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^N a_n.$$

A series is **convergent** if $\lim_{N\to\infty} s_N = s$. Otherwise, it is **divergent**.

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent. A series is **conditionally convergent** if it is convergent but not absolutely convergent.

Riemann's Rearrangement Theorem: A conditionally convergent series can be rearranged to sum to any value.

Basic Series Rules: if $\sum a_n$ and $\sum b_n$ are convergent,

1. $\sum_{n=1}^\infty ca_n = c\sum_{n=1}^\infty a_n$
2. $\sum_{n=1}^\infty (a_n + b_n) = \sum_{n=1}^\infty a_n + \sum_{n=1}^\infty b_n$
3. $\sum_{n=1}^\infty (a_n - b_n) = \sum_{n=1}^\infty a_n - \sum_{n=1}^\infty b_n$.

Notes:

- *ALWAYS check conditions when using convergence tests.*
- *If you're stuck, write out the first few terms.*

Common Types of Series

Geometric series:

$$\sum_{n=0}^\infty ar^n = a + ar + ar^2 + ar^3 + \cdots$$

The geometric series converges for $|r| < 1$ and diverges for $|r| \geq 1$. If $|r| < 1$, the limit is

$$\sum_{n=0}^\infty ar^n = \frac{a}{1-r} \quad (|r| < 1).$$

p-series:

$$\sum_{n=1}^\infty \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

The p-series converges for $p > 1$ and diverges for $p \leq 1$. When $p = 1$, this is the harmonic series.

Telescoping series: A telescoping series has cancellations when adding each new term of the series. Use the partial sum definition to determine convergence.

Example:

$$\sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

$$\begin{aligned} s_1 &= 1 - \frac{1}{2}, \\ s_2 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}, \\ s_3 &= 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}, \\ &\vdots \\ s_N &= 1 - \frac{1}{N+1}. \end{aligned}$$

$$\sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{N\to\infty}\left(1 - \frac{1}{N+1}\right) = 1.$$

Divergence Test

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^\infty a_n$ is divergent.

Note: if $\lim_{n\to\infty} a_n = 0$, then this test is inconclusive!

Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then,

- If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.
- If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

Notes:

- *Don't forget about the continuous, positive, and decreasing assumptions.*
- *To determine if the function is decreasing, find the derivative.*
- *The integral/series need not start at 1: you can start at $n = 2$ or later if needed.*

Comparison Tests

Standard Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is finite and $c > 0$, then either both series converge or both diverge.

Note: don't forget the positive assumption.

Alternating Series Test

If we can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n,$$

where b_n is positive, decreasing, and $\lim_{n\to\infty} b_n = 0$, then $\sum a_n$ converges.

Notes:

- $(-1)^n$ *and* $(-1)^{n+1}$ *are valid.* $(-1)^{2n}$ *is not.*
- $\cos(\pi n) = (-1)^n$.
- $\sin(n)$ *and* $\cos(n)$ *are not alternating.*
- *To determine if b_n is decreasing, find the derivative.*

Ratio and Root Tests

Ratio Test:

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum a_n$ is divergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the test is inconclusive.

Root Test:

- If $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$, then $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} |a_n|^{1/n} = L > 1$, then $\sum a_n$ is divergent.
- If $\lim_{n\to\infty} |a_n|^{1/n} = L = 1$, then the test is inconclusive.

Notes:

- *Don't forget to take the absolute value.*
- *For ratio test, look for factorials. For root test, look for nth powers.*

Power Series

A **power series** centered at $x = a$ is of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

The c_n's are the **coefficients** of the series.

Convergence of Power Series: The **radius of convergence** is a number R such that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. If the series converges only when $x = a$, then $R = 0$. If the series converges for all x, then $R = \infty$. The **interval of convergence** consists of all values of x for which the series converges; this includes the endpoints.

Finding Interval of Convergence: First use the Ratio or Root Test to determine the radius of convergence R. Then, solve for the endpoints using a different convergence test.

Function Representations: use algebraic manipulations (and derivatives/integrals) to switch between functions and their series representations.
Example:

$$\frac{1}{1+3x^2} = \frac{1}{1-(-3x^2)} = \sum_{n=0}^{\infty} (-3x^2)^n = \sum_{n=0}^{\infty} (-3)^n x^{2n}$$

Differentiation and Integration: use the power rule term-by-term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radius of convergence remains the same under differentation and integration, but the convergence of the endpoints can change.

Taylor Series

The **Taylor series** of a function f centered at $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

The **Maclaurin series** is the Taylor series centered at zero. The **Taylor polynomial** $T_n(x)$ of degree n is the partial sum of the Taylor series up to the degree n term.

Taylor's Inequality: if $|f^{(n+1)}(x)| \le M_n$ for $a - d < x < a + d$, then the remainder $R_n(x) = f(x) - T_n(x)$ of the Taylor series satisfies

$$|R_n(x)| \le \frac{M_n}{(n+1)!}|x-a|^{n+1} \quad (\text{for } a-d < x < a+d)$$

Note: to calculate M_n, use the same technique as calculating K described above in "Approximate Integration".

Common Taylor Series

These will be on the front page of the exam.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n \qquad R = 1$$

Differential Equations

A **differential equation** contains an unknown function and one or more of its derivatives. Its **order** is the highest derivative that occurs.

The **general solution** to a differential equation is a family of functions containing one or more arbitrary constants. An **initial-value problem** specifies an initial condition to solve for these constants and obtain one solution.

First-Order Differential Equations

A generic first-order differential equation is

$$y' = F(x, y).$$

An **equilibrium solution** or **constant solution** $y(x)$ solves $y' = F(x, y) = 0$.

The slope $F(x, y)$ can be plotted at each point (x, y). This is called a **direction (slope) field**. Solutions to the differential equation $y(x)$ follow the slopes.

An **autonomous** differential equation is

$$y' = F(y) \quad (\text{does not depend on } x).$$

The graph y' vs. y, known as a *phase portrait*, can be used to create the direction field.

Separable Equations

Separable equations take the form

$$\frac{dy}{dx} = g(x)h(y).$$

To solve these, first find constant solutions satisfying

$$h(y) = 0.$$

Then, solve for non-constant solutions by separating variables:

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx.$$

Note: don't forget $+C$ after the integration step.

Orthogonal Trajectories

To find **orthogonal trajectories** to a family of curves,

1. Implicitly differentiate to find the differential equation $y' = F(x, y)$ that the family of curves satisfies.
2. Solve the differential equation $y' = -\frac{1}{F(x,y)}$ for the orthogonal trajectories.

First-Order Linear Equations

First-order linear equations take the form

$$\frac{dy}{dx} + a(x)y = b(x).$$

To solve these, find the integrating factor:

$$A(x) = e^{\int a(x)\, dx}.$$

Then, the solution is

$$y(x) = \frac{1}{e^{A(x)}} \int e^{A(x)} b(x)\, dx.$$

Note: don't forget $+C$ after the integration step.

Population Growth

Natural Growth: For a population size P, relative growth rate k, and initial population size P_0,

$$\frac{dP}{dt} = kP, \; P(0) = P_0.$$

The solution is

$$P(t) = P_0 e^{kt}.$$

Logistic Growth: For a population size P, relative growth rate k, carrying capacity M, and initial population size P_0,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \; P(0) = P_0.$$

The solution is

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \; A = \frac{M - P_0}{P_0}.$$

Second-Order Linear Differential Equations

A **constant-coefficient, second-order, linear differential equation** takes the form

$$ay'' + by' + cy = g(x).$$

It is **homogeneous** if $g(x) = 0$ and **nonhomogeneous** if $g(x) \neq 0$.

An **initial-value problem** specifies $y(x_0)$ and $y'(x_0)$ at some point x_0. A **boundary-value problem** specifies $y(x_0)$ and $y(x_1)$ at two different points x_0 and x_1.

Homogeneous case:

$$ay'' + by' + cy = 0.$$

To solve, find roots of the **auxiliary equation**

$$ar^2 + br + c = 0.$$

Three cases:

1. Distinct real roots r_1 and r_2:

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x}.$$

2. Repeated real root $r = r_1 = r_2$:

$$y(x) = C_1e^{rx} + C_2xe^{rx}.$$

3. Complex roots $r = \alpha \pm \beta i$:

$$y(x) = C_1e^{\alpha x}\cos(\beta x) + C_2e^{\alpha x}\sin(\beta x).$$

Nonhomogeneous case:

$$ay'' + by' + cy = g(x).$$

To solve, first find the **complementary solution** $y_c(x)$ using the method above. Then, find the **particular solution** $y_p(x)$ using undetermined coefficients or variation of parameters. The general solution is

$$y(x) = y_c(x) + y_p(x).$$

Method of Undetermined Coefficients

For the **method of undetermined coefficients**, we guess the form of $y_p(x)$. There are three key cases:

1. Polynomials: if $g(x)$ is a polynomial, guess a general polynomial of the same degree:

$$g(x) = 3x^2 \implies y_p(x) = Ax^2 + Bx + C.$$

2. Exponentials: if $g(x)$ is an exponential, guess the same exponential with an unknown coefficient:

$$g(x) = 2e^{-4x} \implies y_p(x) = Ae^{-4x}$$

3. Sine/cosine: if $g(x)$ is a sine or cosine, guess a sine and cosine together:

$$g(x) = \cos(2x) \implies y_p(x) = A\cos(2x) + B\sin(2x)$$

If $g(x)$ is a sum, treat each term separately:

$$g(x) = e^{-3x} + \cos(2x)$$
$$\implies y_p(x) = Ae^{-3x} + B\cos(2x) + C\sin(2x)$$

If $g(x)$ is a product, follow steps in this order:

1. Leave the exponential term by itself (if present).
2. Split the sine and cosine (if present).
3. Make polynomial guesses.

$$g(x) = xe^{-x}\cos(2x)$$
$$\implies y_p(x) = e^{-x}[(Ax + B)\cos(2x) + (Cx + D)\sin(2x)]$$

If a term in your guess conflicts with the complementary solution $y_c(x)$, multiply $y_p(x)$ by x or x^2:

$$y_c(x) = C_1\cos(2x) + C_2\sin(2x),\ g(x) = \cos(2x)$$
$$\implies y_p(x) = Ax\cos(2x) + Bx\sin(2x)$$

After making a guess for $y_p(x)$, plug into the differential equation, group like terms together, and solve for the undetermined coefficients.

Series Solutions

A **series solution** to a differential equation takes the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Steps for solving:

1. Find $y'(x)$ and $y''(x)$ and plug into the differential equation.
2. Match the degree and starting indices to combine into one series and solve for the **recursion relation** of the coefficients c_n.
3. Solve for the general term c_n. You will need to leave c_0 (and possibly c_1) undetermined.
4. Write $y(x)$ using c_n. You may need to split the series into several series for even/odd values of n, etc.
5. For initial-value problems, solve for c_0 (and c_1) if necessary.