1. $\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n$ converges.

   Solution: **True.** This is a geometric series with $r = \frac{1}{3} < 1$.

2. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   Solution: **True.** Use integral test.

3. The Maclaurin series of $f(x)$ is the Taylor series of $f(x)$ centered at $x = 0$.

   Solution: **True.** This is the definition of a Maclaurin series.

4. If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

   Solution: **True.** This is a known property of absolute convergence.

5. $\int e^{2x}\, dx = \frac{C}{2} e^{2x}$.

   Solution: **False.** $\int e^{2x}\, dx = \frac{1}{2} e^{2x} + C$.

6. $y' = y - 2$ is a separable and autonomous differential equation.

   Solution: **True.** It is $y' = g(x)f(y)$, where $g(x) = 1$ and $f(y) = y - 2$. It is autonomous since the right-hand side does not depend on x.

7. If the interval of convergence of a power series is $[0, 4)$, then the radius of convergence is $R = 2$.

   Solution: **True.** The radius of convergence is half the length of the interval.

8. $$\int_0^{\pi/2} \sin^5(x) \cos^2(x)\, dx = \int_0^1 (1 - u^2)^2 u^2\, du.$$

   Solution: **True.** This uses the substitution $u = \cos(x)$ together with the trig identity $\sin^2(x) = 1 - \cos^2(x)$.

9. If $f(x) = \sum_{n=0}^{\infty} 2^n (x - 1)^n$, then $f^{(2024)}(1) = 2024! \cdot 2^{2024}$.

   Solution: **True.** This is the Taylor series formula.

10. $\sum_{n=1}^{\infty}(-1)^n n^n$ diverges.

    Solution: **True.** Applying the root test gives n, which diverges to ∞.

11. $\frac{x^2}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$ is a valid partial fraction decomposition.

    Solution: **False.** The degree of the numerator and denominator are both 2, so you must divide first.

12. If the nonhomogeneous part of a second-order, linear, constant-coefficient differential equation is $g(x) = e^x$, then $y_p(x) = Ae^x$ is always a valid form of the particular solution.

    Solution: **False.** If the homogeneous solution contains and e^x term, then you must boost the particular solution by x or x^2.

13. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\cos(\pi n)n^2}$ converges by the alternating series test.

    Solution: **False.** Since $\cos(\pi n) = (-1)^n$, this series is actually equivalent to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is not an alternating series. This is a convergent p-series.

14. The auxiliary equation for $y'' + 2y' = 0$ is $r^2 + 2 = 0$.

    Solution: **False.** The auxiliary equation is $r^2 + 2r = 0$.

15. A boundary-value problem always has exactly one solution.

    Solution: **False.** A boundary-value problem can have infinitely many solutions or no solution.

16. $\int \frac{1}{x^2}\,dx = \ln(x^2) + C$.

    Solution: **False.** $\int \frac{1}{x^2}\,dx = \frac{1}{x} + C$.

17. If the ratio test on $\sum_{n=0}^{\infty} a_n x^n$ says that the power series converges for $\left|x - \frac{1}{3}\right| < 1$, then the radius of convergence is $R = \frac{1}{3}$.

    Solution: **False.** The radius of convergence is 1. The interval of convergence is centered at $x = 1/3$.

18. $\sum_{n=0}^{\infty} \cos(n^2)\frac{3^n}{4^n}$ is absolutely convergent.

    Solution: **True.** Take the absolute value and compare to the geometric series with $r = 3/4$.

19. If $y^{1/2} = \sin^2(x) + C$, then $y = \sin^4(x) + C^2$.

    Solution: **False.** $y = \left(\sin^2(x) + C\right)^2$.

20. If $\sum_{n=0}^{\infty} c_n(-3)^n$ converges, then it is possible that $\sum_{n=0}^{\infty} c_n$ diverges.

    Solution: **False.** $\sum_{n=0}^{\infty} c_n$ must converge since, from the first fact, we know that $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R \geq 3$.

21. $\frac{1}{3-x} = \frac{1}{1-(x/3)} = \sum_{n=0}^{\infty}(x/3)^n$.

    Solution: **False.** In fact, $\frac{1}{3-x} = \frac{1/3}{1-(x/3)} = \frac{1}{3}\sum_{n=0}^{\infty}(x/3)^n$.

22. The Taylor series of $f(x) = e^x$ centered at $x = 2$ is $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$.

    Solution: **False.** It is $e^{x-2+2} = e^2 e^{x-2} = \sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$

23. $\sum_{n=1}^{\infty} n^{4/3}$ converges since $p = 4/3 > 1$.

    Solution: **False.** This is $p = -4/3 < 1$, so the series diverges.

24. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges.

    Solution: **False.** If the limit of the sequence is 0, you cannot say anything. In fact, this is a divergent p-series with $p = 1/2$.

25. The slope field below (Figure 1) describes $y' = 2 - y$.

    Solution: **False.** The point (0,1) should have slope 1.

26. $3x^2y' - 4\ln(x)y = x^2$ is a first-order linear differential equation.

    Solution: **True.** After simplifying, this is of the form $y' + a(x)y = b(x)$.

27. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ converges.

    Solution: **True.** The alternating harmonic series converges by the alternating series test.

28. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}-2}$ converges.

    Solution: **True.** Use the Limit Comparison Test to the p-series with $p = 3/2$.

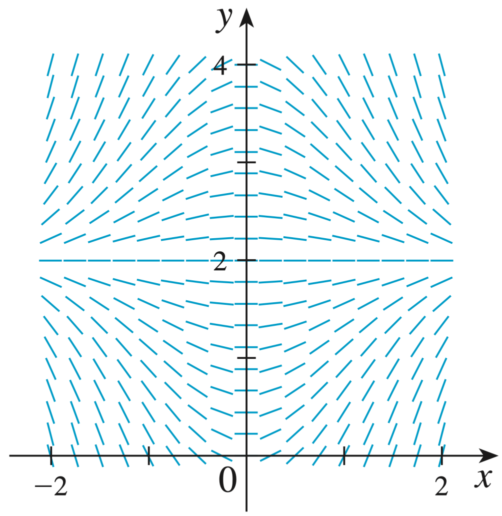

Figure 1: (Credit: Stewart Calculus)

29. If the Ratio Test is inconclusive, then you can still use a different test to determine if the series converges.

    Solution: **True.** If the Ratio Test is inconclusive, you can still try other tests (e.g. comparison, alternating, etc.)

30. $\sum_{n=1}^{\infty}(-1)^n n$ diverges by the Root Test.

    Solution: **False.** The Root Test is inconclusive for this test. The series still diverges by the Divergence Test.

31. The radius of convergence of a power series can be zero.

    Solution: **True.** Consider the power series $\sum_{n=0}^{\infty} n!x^n$.

32. Consider the differential equation $y' = y^2 - 1$. If $y(0) = 0$, then $\lim_{x\to\infty} y(x) = \infty$.

    Solution: **False.** By sketching the phase portrait and/or slope field, you will find $\lim_{x\to\infty} y(x) = -1$.

33. If the complementary solution of a nonhomogeneous, second-order, linear, constant-coefficient differential equation is $y_c(x) = C_1e^{2x} + C_2e^{-x}$ and the nonhomogeneous part is $g(x) = x\cos x$, then a valid guess for a particular solution is $y_p(x) = (Ax + B)(C\cos x + D\sin x)$.

    Solution: **False.** $y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x$.

34. $\int \tan x\, dx = \sec^2 x + C$.

Solution: **False.** $\int \tan x\, dx = \ln|\sec x| + C$.

35. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$.

Solution: **True.** This is the Taylor series expansion of e^x with $x = -1$.

36.

$$\int_0^{\sqrt{10}} \frac{x^3}{\sqrt{10+x^2}}\, dx = \int_0^{\pi/4} \frac{(10^{3/2}\tan^3\theta)(\sqrt{10}\sec^2\theta)}{\sqrt{10+10\tan^2\theta}}\, d\theta = 10^{3/2}\int_0^{\pi/4} \tan^3\theta \sec\theta\, d\theta.$$

Solution: **True.** This is trig substitution with $x = \sqrt{10}\tan\theta$.

37. $(n!)^2 = (2n)!$

Solution: **False.** $(n!)^2$ can't be simplified further.

38. $\sum_{n=0}^{\infty} (-1)^n x^{5n+2}$ is a power series.

Solution: **True.** This is still $\sum_{n=0}^{\infty} c_n x^n$, but most of the c_n's are zero.

39. Since $\lim_{n\to\infty} \frac{3\sqrt{n}+2+n^{-1}}{\sqrt{n}} = 3$, the series $\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}+2+n^{-1}}$ converges because $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges.

Solution: **False.** The series diverges because $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges; it is a p-series with $p = 1/2 < 1$.

40. If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R = 7$, then the series $\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$ has radius of convergence $R = 7$.

Solution: **True.** The derivative of $\sum_{n=0}^{\infty} a_n x^n$ is $\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$, and the radius of convergence of the derivative is the same.

41. $\int_1^{\infty} \frac{1}{x}\, dx$ converges.

Solution: **False.** $\int_1^{\infty} \frac{1}{x}\, dx = \ln|x|\Big|_1^{\infty} = \infty$.

42. $\arccos(x) = \cos^{-1}(x) = \frac{1}{\cos(x)}$

Solution: **False.** $\arccos(x) = \cos^{-1}(x) \neq \frac{1}{\cos(x)} = \sec(x)$.

43. $\frac{1}{x^2(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2}$ is a valid partial fraction decomposition.

Solution: **True.** This satisfies the rules for repeated linear factors and irreducible quadratics.

44. The harmonic series is conditionally convergent.

Solution: **False.** The *alternating harmonic series* is conditionally convergent.

45.
$$(1+x)^{-1/3} = \frac{1}{(1+x)^{1/3}} = \frac{1}{1+x^{1/3}} = \sum_{n=0}^{\infty}(-x^{1/3})^n.$$

Solution: **False.** This is in fact a binomial series with $k = -1/3$.

46. Since $\frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}$ for all $n \geq 1$, the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ converges.

Solution: **True.** These series are both positive, and the standard comparison test holds.

47. If the recursion relation of a series solution $\sum_{n=0}^{\infty} c_n x^n$ to a second-order linear differential equation is $c_{n+2} = \frac{1}{2}c_n$, then the general solution is
$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{1}{2^k} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{1}{2^k} x^{2k+1}.$$

Solution: **True.** The even and odd terms are separate, and writing out the first several terms of the recursion relation yields the above series.

48. $\int \sin(1-x)\,dx = -\cos(1-x) + C$.

Solution: **False.** The substitution $u = 1-x$ means $du = -dx$, so the solution is $\cos(1-x)+C$ (positive sign).

49. $\sum_1^{\infty} \frac{\cos n}{\sqrt{n}}$ converges by the alternating series test.

Solution: **False.** $\cos n$ is not strictly alternating. The alternating series test only works for terms like $(-1)^n, (-1)^{n+1}$, etc.

50.
$$\int \frac{1}{x^2+2x+4}\,dx = \int \frac{1}{(x+1)^2+3}\,dx = \int \frac{1}{u^2+3}\,du \quad \text{where } u = x+1.$$

Solution: **True.** This is the complete-the-square formula.

51. $y' = xy + x$ is a separable differential equation.

    Solution: **True.** This can be re-written as $y' = x(y+1)$.

52. $\sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n-1} \right) = -\frac{3}{2}$.

    Solution: **True.** This is a telescoping series and the only terms that don't cancel are -1 and $-1/2$.

53. If $\sum a_n$ is convergent, then $\sum (-1)^n a_n$ is convergent.

    Solution: **False.** This doesn't work with $a_n = (-1)^n \frac{1}{n}$. This statement is true if a_n is always positive.

54. For the differential equation $y' + 2xy = 3x$, one possible integrating factor is $A(x) = e^{x^2}$.

    Solution: **True.** The integrating factor is $A(x) = e^{\int 2x\,dx}$.

55. $$\cos\left(\arctan\left(\frac{x}{2}\right)\right) = \frac{\sqrt{x^2+4}}{2}.$$

    Solution: **False.** The numerator and denominator are flipped: should be $\frac{2}{\sqrt{x^2+4}}$.

56. The series $\sum_{n=1}^{\infty} n^2 e^{-n}$ converges.

    Solution: **True.** This satisfies all three conditions for the Integral Test (continuous, positive, eventually decreasing) and the integral converges.

57. If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is $R = 2$, then the interval of convergence is $(-2, 2)$.

    Solution: **False.** The endpoints could be included in the interval.

58. All solutions to the initial-value problem $y' = \sqrt{y}$ are of the form $y = \left(\frac{1}{2}x + C\right)^2$.

    Solution: **False.** $y = 0$ is also a solution.

59. If $\sum_{n=0}^{\infty}(-1)^n c_n 2^n$ is convergent, then $\sum_{n=0}^{\infty} \frac{c_n}{4^n}$ is convergent.

    Solution: **True.** By the given statement, the power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = \frac{1}{2}$, so $R \geq \frac{1}{2}$. Therefore, $x = \frac{1}{4}$ lies in the interval of convergence.

60. If $f(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{n+2}$, then $f^{(500)}(0) = 3^{500}$.

    Solution: **False.** Re-index first to see $f(x) = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} x^n$. Thus, $f^{(500)}(0) = 3^{498}$.